LAGRANGE EQUATIONS FOR A SYSTEM OF BUBBLES IN A LIQUID OF LOW VISCOSITY

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We attempt the description of the motion of a system of bubbles in a low viscosity liquid by means of Lagrange equations with the kinetic energy of an ideal liquid as the Lagrangian function. The acceleration of each bubble is assumed to be small such that the change in velocity) can be neglected. A consideration of the virtual power of the generalized external forces leads to an expression, differing from that in [1], for the viscous friction force acting on a bubble in the system. The effect of the concentration and form of the system on the velocity of ascent of the bubbles for low bubble concentration is investigated. The maximum size of the region occupied by the bubbles when the effect of wakes on the motion of the system can be neglected is estimated.

1. Lagrange Equations. The motion of a system of gas bubbles in an ideal incompressible fluid can be described [2] by Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}_i} - \frac{\partial T}{\partial \mathbf{r}_i} = \mathbf{Q}_i \qquad (1.1)$$

with a Lagrangian function equal to T, the kinetic energy of an ideal liquid. Here T is determined by the radius-vectors $\mathbf{r}_i$ of the centers of the bubbles and their velocities $\mathbf{u}_i$ at the same instant ($i = 1, \ldots, N$), and $\mathbf{Q}_i$ is the generalized external force acting on the i-th bubble of the system. Such a description is possible since the interaction propagates with the speed of sound, which can be regarded as infinite for an incompressible medium. In fact, in the case of the potential flow of an ideal liquid the velocity of the liquid at any point at a given instant is determined by the velocities of the moving bubbles at the same instant.

In the case of a system of bubbles of moderate size in a liquid of low viscosity ($1 \ll R < 300$, $R = ua/\nu$ is the Reynolds number, and u is the velocity of a bubble of radius a in a liquid with kinematic viscosity) the sum of all the forces acting on the i-th bubble ($i = 1, \ldots, N$) of negligibly small mass will be zero and the only force acting on such a bubble in the liquid will be the hydrodynamic pressure

$$\int(-pn_i^\alpha + \sigma'^{\alpha\beta} n_i^\beta)\, dS = \int(-p + \sigma_i'^{rr})\, n_i^\alpha\, dS = 0$$

$$\sigma'^{\alpha\beta} = \mu\,(\partial v^\alpha / \partial r^\beta + \partial v^\beta / \partial r^\alpha), \quad \sigma'^{rr} = 2\mu\, \partial v^r / \partial r \qquad (1.2)$$

where p is the pressure in the liquid, n_i^α are the components of the unit vector of the external normal to dS—an element of surface of the i-th bubble, $\sigma'^{\alpha\beta}$ are the components of the viscous stress-tensor with a single nonzero component $\sigma'^{\gamma\gamma}$ in a spherical coordinate system with its center at the center of the i-th bubble, μ is the dynamic viscosity, ν^α are the components of the liquid velocity, and r^α are the coordinates of the observation point ($\alpha = 1, 2, 3$). In (1.2) and henceforth we assume summation over the repeated Greek indices.

The pressure p can be represented as the sum of p_0—the pressure in an ideal liquid—and p'—an additional viscosity-dependent term, which in conjunction with $\sigma_i'^{rr}$ leads to the appearance of a viscous friction force. For an ideal liquid [3]

$$-\int p_0 \mathbf{n}_i\, dS = \frac{d}{dt}\frac{\partial T}{\partial \mathbf{u}_i} - \frac{\partial T}{\partial \mathbf{r}_i} \qquad (1.3)$$

and hence, by including the viscous friction force in the external forces acting on the i-th bubble we can regard the Lagrange equations as a corollary of (1.2) and (1.3).

However, a consideration of the viscosity-dependent forces means that if in the description of the motion of a system of bubbles in a viscous liquid we describe the system not by considering the interaction of each bubble with the surrounding liquid, but as an assembly of material points interacting in accordance with a definite law and situated in an external field, then we cannot consider such action as instantaneous. In fact, the viscous friction-force depends on the flow velocity relative to the bubble at the considered instant and on the acceleration of the relative motion at earlier times. A boundary layer with transverse dimensions of the order of $aR^{-1/2}$ is formed around each bubble. The propagation time of the perturbation through the region occupied by the boundary layer is on the order of $a^2/\nu R \sim a/u$. Hence, the change in the velocity of the flow impinging on the bubble leads to a change in the viscous friction-force with a delay on the order of a/u. The velocity of the impinging flow on the considered bubble depends on the velocities of the other bubbles in the system. It follows that the viscous friction force includes the particle interaction force propagated with finite velocity.

Such interaction cannot be included in the Lagrangian function, which depends only on the coordinates and velocities at the considered instant nor in the generalized external forces $\mathbf{Q}_i$, since by definition the $\mathbf{Q}_i$ entail the assumption that the liquid velocity at any point is determined by the instantaneous positions and velocities of the bubbles.

The finiteness of the propagation velocity of the interaction makes it impossible to give a rigorous description of the motion of a system of bubbles in a viscous liquid by means of Lagrange equations with a Lagrangian function equal to the kinetic energy of an ideal liquid. To determine the generalized external forces $\mathbf{Q}_i$ for bubbles in a low viscosity liquid we follow Levich [4] and assume that the velocity field around the bubbles does not differ greatly from the velocity field of an ideal liquid. The corrections to the kinetic energy of the liquid flowing around a single bubble will be of the order of 1/R.

By repeating the arguments of [2] we show that for a low viscosity liquid the generalized forces will be the coefficients in front of the virtual bubble velocities Dr_i^α/Dt in the expression for the virtual power of the forces acting on the system, which, in view of (1.2), is equal to the virtual power of the forces acting on the liquid

$$\int \rho \frac{dv^\alpha}{dt} V^\alpha\, d^3r = \sum_{i=1}^{N} Q_i^\alpha \frac{Dr_i^\alpha}{Dt} \qquad (1.4)$$

where ρ is the density of the liquid, V^α is the component of the virtual liquid velocity due to the virtual motion of the bubbles with velocities Dr_i^α/Dt.

We consider the motion of a system of bubbles of radius a. The bubble velocities relative to the liquid at rest at infinity are u_i. The velocity field potential of this liquid satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{1.5}$$

with boundary conditions

$$n_i^\alpha \partial\Phi / \partial r^\alpha = n_i^\alpha u_i^\alpha \quad \text{at} \quad r_i' = a, \text{ and}$$

$$\Phi \to 0 \quad \text{at} \quad r_i' \to \infty$$

$$(r_i' = |\mathbf{r}_i'|, \quad \mathbf{r}_i' = \mathbf{r} - \mathbf{r}_i, \quad i = 1, \ldots, N). \tag{1.6}$$

The solution of this problem, accurate to terms of order $(a/l)^3$, where l is the mean distance between the bubble centers is

$$\Phi = -\frac{a^3}{2}\sum_{i=1}^{N}\frac{v_i^\alpha r_i'^\alpha}{r_i'^3},$$

$$v_i^\alpha = u_i^\alpha - \frac{a^3}{2}\sum_{j=1}^{N} u_j^\beta \Lambda_{ji}^{\beta\alpha}, \quad \Lambda_{ii}^{\alpha\beta} \equiv 0,$$

$$\Lambda_{ij}^{\alpha\beta} = \frac{3r_{ij}^\alpha r_{ij}^\beta}{r_{ij}^5} - \frac{\delta^{\alpha\beta}}{r_{ij}^3} \quad (i \neq j), \quad \delta^{\alpha\beta} = \begin{cases}1 & \alpha = \beta)\\ 0 & \alpha \neq \beta)\end{cases},$$

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \quad r_{ij} = |\mathbf{r}_{ij}| \tag{1.7}$$

The virtual displacements of the bubbles give rise to a virtual potential

$$\Phi' = -\frac{a^3}{2}\sum_{j=1}^{N}\varphi_i^\alpha \frac{Dr_i^\alpha}{Dt}$$

$$\left(\varphi_i^\alpha = \frac{r_i'^\alpha}{r_i'^3} - \frac{a^3}{2}\sum_{j=1}^{N}\Lambda_{ij}^{\alpha\beta}\frac{r_j'^\beta}{r_j'^3}\right)$$

corresponding to the virtual velocity of the liquid

$$\frac{Dr^\alpha}{Dt} = -\frac{a^3}{2}\sum_{i=1}^{N}\frac{\partial\varphi_i^\beta}{\partial r^\alpha}\frac{Dr_i^\beta}{Dt}.$$

Thus, the generalized external forces are defined as

$$Q_i^\alpha = -\frac{a^3}{2}\int \rho \frac{dv^\beta}{dt}\frac{\partial\varphi_i^\alpha}{\partial r^\beta} d^3r. \tag{1.8}$$

In accordance with the Navier-Stokes equations for an incompressible viscous fluid

$$\rho\, dv^\alpha / dt = -\partial p / \partial r^\alpha + \partial\sigma'^{\alpha\beta} / \partial r^\beta + \rho g^\alpha \tag{1.9}$$

where g is the gravitational acceleration, Eq. (1.8) can be transformed to

$$Q_i^\alpha = \frac{a^3}{2}\int\frac{\partial}{\partial r^\beta}\left(p\frac{\partial\varphi_i^\alpha}{\partial r^\beta} - \sigma'^{\gamma\beta}\frac{\partial\varphi_i^\alpha}{\partial r^\gamma}\right)d^3r +$$

$$+\frac{a^3}{2}\int\sigma'^{\gamma\beta}\frac{\partial^2\varphi_i^\alpha}{\partial r^\beta\partial r^\gamma}d^3r - \frac{\rho a^3}{2}\int\frac{\partial}{\partial r^\beta}g^\gamma r^\gamma\frac{\partial\varphi_i^\alpha}{\partial r^\beta}d^3r \tag{1.10}$$

if we use the obvious identity $\Delta\varphi_i^\alpha \equiv 0$ when $r_i' \neq 0$ ($i = 1, \ldots, N$). In addition, it follows from the definition of φ_i^α that on the surface of the j-th sphere:

$$n_j^\beta \frac{\partial\varphi_i^\alpha}{\partial r^\beta} = -2\frac{n_i^\alpha}{a^3}\delta_{ij}, \quad \delta_{ij} = \begin{cases}1 & (i = j)\\ 0 & (i \neq j)\end{cases}. \tag{1.11}$$

Hence, we can rewrite Q_i^α, using $\sigma'^{\gamma\beta} = \sigma'^{rr} n_j^\gamma n_j^\beta$ and (1.11), in the form

$$Q_i^\alpha = \int(p - \sigma'^{rr})\, n_i^\alpha\, dS +$$

$$+\frac{a^3}{2}\int\sigma'^{\gamma\beta}\frac{\partial^2\varphi_i^\alpha}{\partial r^\beta\partial r^\gamma}d^3r - \frac{4\pi}{3}\rho a^3 g^\alpha. \tag{1.12}$$

The integral over the infinitely distant surface bounding the system disappears, since it is assumed that the region occupied by the bubbles has a finite size and, hence, when $r \to \infty$ we obtain $p \to$ const, $\partial\varphi_i^\alpha/\partial r^\beta \sim r^{-3}$, and $\sigma'^{\sigma\beta} \sim r^{-4}$.

The first integral in (1.12), in accordance with (1.2) is zero, the second term is the Archimedes force, and the third term is the viscous friction force, which can be calculated with an error of the order of $R^{-1/2}$ by replacing the velocity field in the whole space by the velocity field of an ideal liquid [5]

$$\frac{a^3}{2}\int\sigma'^{\gamma\beta}\frac{\partial^2\varphi_i^\alpha}{\partial r^\beta\partial r^\gamma}d^3r = \mu a^3\int\frac{\partial^2\Phi}{\partial r^\beta\partial r^\gamma}\frac{\partial^2\varphi_i^\alpha}{\partial r^\beta\partial r^\gamma}d^3r =$$

$$= -\mu a^3\sum_{j=1}^{N}\int\frac{\partial^2\Phi}{\partial r^\beta\partial r^\gamma}n_j^\beta\frac{\partial\varphi_i^\alpha}{\partial r^\gamma}dS.$$

It follows from (1.7) and the definition of φ_i^α that on the surface of the j-th sphere:

$$an_j^\beta\frac{\partial^2\Phi}{\partial r^\beta\partial r^\gamma} = -{}^3/_2 v_j^\beta(3n_j^\beta n_j^\gamma - \delta^{\beta\gamma}),$$

$$\frac{\partial\varphi_i^\alpha}{\partial r^\gamma} = -\frac{\delta_{ij}}{a^3}(3n_j^\alpha n_j^\gamma - \delta^{\alpha\gamma}) + \frac{3}{2}\Lambda_{ij}^{\alpha\beta}(n_j^\beta n_j^\gamma - \delta^{\beta\gamma}).$$

Thus, we obtain

$$Q_i^\alpha = -\frac{4\pi}{3}\rho a^3 g^\alpha - 12\pi\mu a v_i^\alpha + 6\pi\mu a^4\sum_{j=1}^{N}u_j^\beta\Lambda_{ji}^{\beta\alpha}. \tag{1.13}$$

The viscous friction force acting on a bubble in the system consists of two terms, the first of which corresponds to a force, given by Levich's formula [4], acting on a single bubble on which a flow with velocity $-\mathbf{v}_i$ impinges. The velocity of the liquid in a system centered at the i-th bubble, in its vicinity, and as follows from (1.7), is such that in the problem of a stationary single bubble washed by a flow with velocity $-\mathbf{v}_i$ at infinity

$$\frac{\partial\Phi}{\partial r^\alpha} - u_i^\alpha = \frac{a^3}{2}\left(\frac{3v_i^\beta r_i'^\beta r_i'^\alpha}{r_i'^5} - \frac{v_i^\alpha}{r_i'^3}\right) - v_i^\alpha.$$

The last term in formula (1.13) appears because the viscous friction force depends not only on the velocity of the flow impinging on the bubble, but also on the effective width of the wake, which in turn depends on the coordinates and velocities of other bubbles in the system. The special case presented in 4 confirms (1.13). Calculations of the viscous friction force as the momentum unit time transmitted from the bubble to the liquid, and based on the transport in the wake of the vortex formed in the boundary layer, agree with those done by the energy dissipation method, as shown by Moore [5] for the steady motion of a single bubble.

The kinetic energy of an ideal liquid flowing around a system of spheres is given up to order $(a/l)^3$ [1] by

$$T = \frac{\pi\rho a^3}{3}\Big(\sum_{i=1}^{N} u_i^2 - \frac{3a^3}{2}\sum_{i,j=1}^{N} u_i^\alpha \Lambda_{ij}^{\alpha\beta} u_j^\beta\Big). \qquad (1.14)$$

Since the kinetic energy (1.14) is a homogeneous quadratic function of the velocities $\mathbf{u}_i$

$$\sum_{i=1}^{N} \mathbf{u}_i \frac{\partial T}{\partial \mathbf{u}_i} = 2T \qquad (1.15)$$

then from the Lagrange equations (1.1) with generalized external forces (1.13) it follows that

$$\frac{d}{dt} T = -\frac{4\pi}{3}\rho a^3 \sum_{i=1}^{N} (\mathbf{g}\cdot\mathbf{u}_i) - 12\pi\mu a \sum_{i=1}^{N} v_i^2. \qquad (1.16)$$

We can convince ourselves of the correctness of this result by using the Navier-Stokes equations to calculate the change in energy of low viscosity liquid where the velocity field differs little from that of an ideal liquid.

Thus, the Lagrange equations for the motion of a system of bubbles in a low viscosity liquid are

$$\frac{2\pi\rho a^3}{3}\frac{d}{dt}\Big(u_i^\alpha - \frac{3a^3}{2}\sum_{j=1}^{N} u_j^\beta \Lambda_{ji}^{\beta\alpha}\Big) + \pi\rho a^6 \sum_{j=1}^{N} u_i^\beta \frac{\partial \Lambda_{ij}^{\beta\gamma}}{\partial r_{ij}^\alpha} u_j^\gamma = -\frac{4\pi}{3}\rho a^3 g^\alpha - 12\pi\mu a\Big(u_i^\alpha - a^3\sum_{j=1}^{N} u_j^\beta \Lambda_{ji}^{\beta\alpha}\Big). \qquad (1.17)$$

Neglecting the terms containing the small parameter $(a/l)^3$, we obtain the equations of motion of an individual bubble

$$\frac{d\mathbf{u}_i}{dt} = -2\mathbf{g} - \frac{18\nu}{a^2}\mathbf{u}_i. \qquad (1.18)$$

If this expression is substituted in the terms of the left side of Eqs. (1.17), containing the small parameter, then up to terms of order $(a/l)^3$ the Lagrange equations are

$$\frac{du_i^\alpha}{dt} = \frac{3a^3}{2}\sum_{j=1}^{N} u_j^\beta \frac{\partial \Lambda_{ji}^{\beta\gamma}}{\partial r_{ji}^\alpha} u_j^\beta - 2\Big(g^\alpha + \frac{3a^3}{2}\sum_{j=1}^{N} g^\beta \Lambda_{ji}^{\beta\alpha}\Big) - \frac{18\nu}{a^2}\Big(u_i^\alpha + \frac{a^3}{2}\sum_{j=1}^{N} u_j^\beta \Lambda_{ji}^{\beta\alpha}\Big) \qquad (1.19)$$

The derivation of (1.19) entails the use of the symmetry of the tensor $\Lambda_{ij}^{\alpha\beta}$

$$\partial \Lambda_{ij}^{\alpha\beta} / \partial r_{ij}^\gamma = \partial \Lambda_{ij}^{\alpha\gamma} / \partial r_{ij}^\beta .$$

The Lagrange equations (1.19) are suitable for the description of processes with sufficiently small accelerations $|du_i/dt| \ll u_i^2/a$. By using these equations we can, for instance, solve the problem of motion of a single bubble of a size corresponding to the Reynolds number $R \gg 18$, since in this case the relaxation time of the steady state velocity $a^2/18\nu$ significantly exceeds the relaxation time of the boundary layer a/u.

2. System of bubbles in an unbounded liquid with spatially homogeneous and isotropic distribution. If the system of bubbles has the form of an ellipsoid, then the sums contained on the right side of equations (1.19) can be expressed by the Lorentz method [6], in terms of the volume concentration of bubbles in the system $c = (4/3)(a/l)^3$ and the depolarizing factor n_z of the ellipsoid ($n_z = 0$ for a long circular cylinder, $n_z = 1/3$ for a sphere, and $n_z = 1$ for a thin plate)

$$a^3 \sum_{j=1}^{N} \Lambda_{ji}^{\alpha\beta} = (1 - 3n_z) c\delta^{\alpha\beta}. \qquad (2.1)$$

Replacing $\mathbf{u}_j$ on the right-hand side of Eq. (1.19) by $\mathbf{u}_0 = -\mathbf{g}a^2/9\nu$, the steady state velocity of ascent of a single bubble of radius a, we obtain the equation of motion of the i-th bubble with the averaged values of the forces acting on it

$$\frac{a^2}{18\nu}\frac{d\mathbf{u}_i}{dt} = \mathbf{u}_0 - \mathbf{u}_i + (1 - 3n_z)c\mathbf{u}_0. \qquad (2.2)$$

The velocity of steady ascent of a bubble in the system, as (2.2) shows, is

$$\mathbf{u} = [1 + (1 - 3n_z) c]\,\mathbf{u}_0. \qquad (2.3)$$

This result differs from the corresponding one obtained in [1], since there it was erroneously assumed that the viscous friction force depends on the concentration.

The ascent of a bounded layer of bubbles in a cylindrical column of liquid in the absence of continuous delivery of gas through the base of the column corresponds, as was shown in [1], to the choice of $n_z = 1$. The velocity of ascent of the bubbles in the column will be $u = u_0(1 - 2c)$. The flow impinging on each bubble will move relative to the tube walls with velocity

$$u_i^\alpha - v_i^\alpha = \frac{a^3}{2}\sum_{j=1}^{N} u_0^\beta \Lambda_{ji}^{\beta\alpha} = \frac{1 - 3n_z}{2} c u_0^\alpha = -c u_0^\alpha .$$

Thus, in the region occupied by the bubbles there is a descending flow of liquid with a mean velocity $-c\mathbf{u}_0$.

We now consider a continuous flow of bubbles through the liquid in a vertical tube with constant delivery of air through the base. It is obvious that the mean liquid velocity over any cross section of the column in this case will be zero. This means that the entering gas imparts to the liquid an additional velocity, which compensates the velocity of descending motion of the liquid, equal to $-c\mathbf{u}_0$. Hence, the velocity of ascent of the bubbles relative to the tube walls will be greater by an amount $c\mathbf{u}_0$ than in the previously considered case and will be $\mathbf{u} = (1 - c_0)\mathbf{u}_0$.

3. Effect of wakes on the motion of a system of bubbles. We consider the motion of a single bubble ($R \gg 1$) in an unbounded liquid. From the known expression for the drag force, equal to $-12\pi\mu a u_i$, we write the velocity field of the liquid far from the bubble in the region outside the wake in the form (Landau and Lifshits [7])

$$\mathbf{v} = 3\nu a \mathbf{r}'_i / r'^3_i . \quad (3.1)$$

In the case of motion of a system of particles the effect of the particle wakes leads to the appearance of an additional velocity of impinging flow in the vicinity of the i-th bubble, equal to

$$\mathbf{v}_i = -3\nu a \sum_{j=1}^{N} \frac{\mathbf{r}_{ij}}{r_{ij}^3} \quad (3.2)$$

which produces additional forces of mutual repulsion on the bubbles.

This correction to the velocity field can be neglected in comparison with the previously calculated velocity of impinging flow (1.7), if

$$\nu a \Sigma \ll u_0 \left(\Sigma = \left|\sum_{j=1}^{N} \frac{\mathbf{r}_{ij}}{r_{ij}^3}\right|\right). \quad (3.3)$$

For a system of bubbles occupying a region of finite length of order L, as estimates show, $\Sigma \sim L/l^3$, then for (3.3) implies

$$L / a \ll R \, (l / a)^3. \quad (3.4)$$

A similar estimate can be obtained in the case of a system of bubbles with homogeneous density occupying an infinite horizontal layer of thickness H. It follows from the solution of the electrostatic problem of a field created by a homogeneously charged layer at $\Sigma = 2\pi z_i / l^3$ (z_i is the distance from the center of the i-th sphere to the middle of the layer).

The velocity of the impinging flow on the i-th bubble will be greater in the upper half of the layer (and less, in the lower half) by $6\pi\nu a z_i/l^3$ than v_i given by (1.7). Thus, the ascending layer of bubbles tends to expand, but will retain its spatially homogeneous distribution. This result is valid as long as the thickness of the layer is sufficiently small, i. e., as long as the probability of a bubble intersecting the wake of any other bubble can be neglected.

We consider the velocity field in the laminar wake of a single bubble on which a flow with velocity U impinges. At large distances from the bubble the velocity component of the liquid v_x along the direction of the impinging flow is

$$v_x = -3U \frac{a}{x} \exp\left(-\frac{U}{4\nu} \frac{y^2+z^2}{x}\right) \quad (3.5)$$

(x, y, and z are the components of the vector r'_i); and v_y and v_z have similar forms. The velocity field in the wake is described by (3.5) only at distances $x \gg aR^{1/2}$, as Moore [5] showed, the width of the wake is virtually constant and is equal to $aR^{-1/4}$, and the stream lines for the laminar part of the wake differ little from the stream lines of an ideal liquid.

The additional velocity field leads to entrainment of bubbles into the wake region. The effect width of the region occupied by the wake, as (3.5) shows, is of the order of $(\nu x/U)^{1/2}$. This is less than the mean distance between the bubbles, if the thickness H of the layer occupied by the bubbles satisfies the inequality

$$H / a \ll R \, (l / a)^2. \quad (3.6)$$

Thus, the effect of wakes on the motion of the system of bubbles can be ignored if the thickness of the layer satisfies (3.6), while in the case of motion of a system of bubbles occupying a region of finite dimensions in an unbounded liquid it is also essential that the horizontal dimensions of the region satisfy inequality (3.4).

4. Viscous friction force in the motion of two bubbles. Let the radii of the bubbles be a, and the distance l between their centers satisfy the condition $a \ll l \ll aR^{1/2}$. Then the velocity of the bubbles (U) is equal and directed along the direction of motion of a single bubble, taken as the z-axis—the polar axis of a spherical system of coordinates, whose origin is at the center of the first bubble. Let the coordinates of the center of the second bubble be $(l, \theta_0, 0)$, where

$$(a / l) \, R^{-1/4} \ll \pi - \theta_0 \ll 1.$$

When these conditions are satisfied, the flow around the first bubble can be regarded as axisymmetric and the wake from the first bubble does not enter the boundary layer of the second. The wakes from such bubbles do not overlap at a distance of the order of $aR^{1/2}$. If the velocity of the impinging flow at infinity is U, the velocity of the impinging flow in the vicinity of the bubbles, according to (1.7), is $U[1 - (a/l)^3]$. The stream function in the vicinity of the first bubble at a distance $r \ll l$ from it is

$$\psi = (U/2) \, (1 - a^3 / l^3) \, (r^2 - a^3/r) \sin^2\theta \quad (4.1)$$

and far from the first bubble for $l \ll r \leqslant R^{1/2}$ it is

$$\psi = (U / 2) \, r^2 \sin^2\theta . \quad (4.2)$$

We draw two planes $S(z = z_1, z_1 \gg l)$ and $S'(z = z_2, l \ll -z_2 \leqslant aR^{1/2})$ so that the current lines intersecting them might be regarded as parallel. The drag force D_1 is the momentum flux imparted to the bubble by the liquid [7]. For the quasi-steady motion under consideration, the drag force is equal to the difference in momentum fluxes across the planes S and S'

$$D_1 = \left(\int_S - \int_{S'}\right) [p + \rho U (v_z - U)] \, dS. \quad (4.3)$$

We use the Bernouilli equation

$$p + (\rho / 2) \, U^2 + \rho \, U \, (v_z - U) + C(\psi) = 0 \quad (4.4)$$

where the function $C(\psi) = 0$, except for the region occupied by the wake, for which calculations similar to those made in [5], lead, with the aid of (4.1), to the following result

$$C(\psi) = 6\sqrt{2}\rho U^2 \left(1 - \frac{a^3}{l^3}\right) \delta f(t), \quad (4.5)$$

$$\delta^2 = \nu \left[aU\left(1 - \frac{a^3}{l^3}\right)\right]^{-1},$$

$$f(t) = (1 / \sqrt{\pi}) \exp(-t^2) - t \operatorname{erfc} t,$$

$$t = \psi \left[2\sqrt{2}\delta a^2 U\left(1 - \frac{a^3}{l^3}\right)\right]^{-1}. \qquad (4.5)$$
(cont'd)

Thus, to calculate the viscous friction force acting on the first bubble we can confine ourselves to integration in the plane S' over the region occupied by the wake from the first bubble. Since the integrand decreases rapidly

$$D_1 = 2\pi \int_0^\infty C(\psi)\, y\, dy \quad (y = r \sin\theta) \qquad (4.6)$$

where, according to (4.2), $d\psi = Uydy$. Hence, it follows that

$$D_1 = 12\pi\mu aU\,(1 - a^3 / l^3)^2. \qquad (4.7)$$

The result obtained confirms the validity of formula (1.13).

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